Deterministic Approximation for the Volume of the Truncated Fractional Matching Polytope

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We study algorithms that *approximately* **compute** volume

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Time: $\mathrm{poly}(\frac{1}{\epsilon}) \cdot \mathrm{poly}(\#\textit{queries}(K))$

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- are fully polynomial-time randomized approximation schemes (FPRAS)

Quest for deterministic counting algorithms

correlation decay	#independent sets #matchings	[Weitz05] [Bayati-Gamarnik-Katz-Nair-Tetali07]
Barvinok's polynomial interpolation method	#independent sets #q-colorings	[Barvinok15] [Patel-Regts17] [Liu-Srivastava-Sinclair19] [Bencs-Berrekkal-Regts24]
cluster expansion	#independent sets (dense random bipartite graphs)	[Helmuth-Perkins-Regts19] [Jensen-Keevash-Perkins20]
local CLTs	#independent sets #matchings	[Jain-Perkins-Sah-Sawhney21]
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Can deterministic algorithms approximate volume?

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Do there exist poly-time deterministic algorithms for nicely-representable convex bodies?

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- are exact algorithms, achieve exponentially-small error!

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Can we leverage combinatorial structure of input polytope to deterministically approximate volume in polynomial-time?

Prior work: fractional independent set polytope

Definition (fractional independent set polytope)

Given graph $G = (V, E)$, the *fractional independent set polytope* is defined as

$$I_G := \left\{ \mathbf{x} \in [0,1]^V \mid x_u + x_v \leq 1 \text{ for every } uv \in E \right\}$$

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- [Gamarnik-Smedira23] gave deterministic quasi-polynomial time algorithm

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- note: volume of polytope and dual can be quite different

Truncated fractional matching polytope

- restrict each variable to interval $M_\delta = \left[0, \frac{1+\delta}{\Delta}\right]$, then $P_{G,\delta} = {M_\delta}^E \cap P_G$:

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- multiplicative relaxation of trivial truncation interval $\left[0, \frac{1}{\Delta}\right]$ ($P_{G,0} \equiv |E|$-dimensional cube)
- let C_v is satisfied $\Leftrightarrow \sum_{e \sim v} x_e \leq 1$

$$\text{Observe } \mathrm{Vol}(P_{G,\delta}) := \int_{{M_\delta}^E} \prod_{v \in V} \mathbb{1}_{C_v} \, d\mu$$

Our main theorem for graph polytope volume

Theorem (truncated fractional matching polytope)

For any graph G *of maximum degree* $\Delta \geq 2$ *and* $\delta \leq \frac{C}{\Delta}$ *for some constant* $C > 0$*,*

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- technique we use: cluster expansion

Polymer models

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Definition (polymer model partition function)

$$\Xi(\mathcal{P}, w) := \sum_{\Gamma} \prod_{\gamma \in \Gamma} w(\gamma),$$

where the sum is over compatible $\Gamma \subseteq \mathcal{P}$

Cluster expansion

Definition (cluster expansion)

cluster expansion of Ξ is the infinite power series:

$$\log \Xi(\mathcal{P}, w) := \sum_{\Gamma \in \mathcal{C}} \Phi(\Gamma) \prod_{\gamma \in \Gamma} w(\gamma)$$

where $\mathcal{C}$ is the set of "clusters"

recall $\Xi(\mathcal{P}, w) := \sum_{\Gamma} \prod_{\gamma \in \Gamma}$

Convergence of cluster expansion

Proposition (Kotecký-Preiss criterion [KP86])

Let $g : \mathcal{P} \to [0, \infty)$ be a "decay function". Suppose that for all $\gamma \in \mathcal{P}$, we have,

$$\sum_{\gamma' \not\sim \gamma} |w(\gamma')| \, e^{|\gamma'| + g(\gamma')} \leq |\gamma|$$

Then, the cluster expansion converges absolutely

Algorithms from cluster expansion convergence

Proposition ([Jenssen-Keevash-Perkins20])

Fix an integer $\Delta > 0$ and let $\mathcal{G}$ be a class of graphs of maximum degree at most Δ. Suppose the following conditions hold for a given polymer model with decay function $g(\cdot)$:

(1) *there exists $\rho > 0$ such that for every $G \in \mathcal{G}$ and every polymer $\gamma \in \mathcal{P}(G)$, $g(\gamma) \geq \rho|\gamma|$*

(2) *there exist constants $c_1, c_2 > 0$ such that given a connected subgraph γ, determining whether γ is a polymer, and then computing $w(\gamma)$ and $g(\gamma)$ can be done in time $O(|\gamma|^{c_1} e^{c_2|\gamma|})$*

(3) *the Kotecký-Preiss criterion is satisfied*

Then there exists an FPTAS for $\Xi(G)$ for every $G \in \mathcal{G}$

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- [Jenssen-Keevash-Perkins20] proof uses Barvinok's interpolation method

Key insight: volume is polymer model partition function

$$\mathsf{Vol}(P_{G,\delta}) = \int_{M_\delta{}^E} \prod_{v \in V} \mathbb{1}_{C_v} d\mu$$
$$= \dots$$
$$= \left(\frac{1+\delta}{\Delta}\right)^{|E|} \sum_{S \subseteq V} \prod_{K \in \mathcal{K}(S)} w(K)$$

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- $w(S) = (-1)^{|S|} \int_{{M_\delta}^{E(S)}} \prod_{v \in S} \mathbb{1}_{\overline{C_v}} d\mu$

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- [Jenssen-Keevash-Perkins20] algorithm: convergence implies using Barvinok's interpolation method we can truncate the partition function up to logarithmically-many terms for a good approximation
- [Jenssen-Keevash-Perkins20] also requires weights of some polymers can be computed efficiently; but those conditions are easy to verify

Verifying Kotecký-Preiss criterion

$$|w(S)| = \int_{M_\delta{}^{E(S)}} \prod_{v \in S} \mathbb{1}_{\overline{C_v}} d\mu$$

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- pick maximal independent set on the vertices, so sets of constraints are independent

$$|w(S)| = \Pr\left[\bigwedge_{v \in S} \overline{C_v} \right] \leq \Pr\left[\bigwedge_{v \in I_S} \overline{C_v} \right] = \prod_{v \in I_S} \Pr\left[\overline{C_v} \right]$$

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- small truncation interval implies probability of violation of a set of constraints decays exponentially in its size

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H with maximum degree Δ and maximum hyperedge size k

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- using similar polymer model as before:

$$\mathrm{Vol}(P_{H,\delta}) = \sum_{S \subseteq V} \int_{{M_\delta}^{E \setminus E(S)}} 1 d\mu \left(\prod_{K \in \mathcal{K}(S)} (-1)^{|K|} \int_{{M_\delta}^{E(K)}} \prod_{v \in K} \mathbb{1}_{\overline{C_v}} d\mu \right)$$

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Theorem ((weaker) fractional hypergraph matching polytope)

For hypergraphs of maximum degree $\Delta \geq 2$ and maximum hyperedge size k and truncation parameter $\delta \leq \Theta\left(\frac{1}{\Delta k}\right)^{k-1}$, there is a fully polynomial-time approximation scheme (FPTAS) for $\mathrm{Vol}(P_{H,\delta})$

Hypergraph matching polytope

Theorem (fractional hypergraph matching polytope)

For hypergraphs of maximum degree $\Delta \geq 2$ and maximum hyperedge size $k \geq 2$, and $\delta \leq \frac{C}{\Delta^{\frac{2k-3}{k-1}} k}$ for some constant $C > 0$, there is an FPTAS for $\mathrm{Vol}(P_{H,\delta})$

- improvement over truncation parameter $\delta \leq \Theta\left(\frac{1}{\Delta k}\right)^{k-1}$

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- improvement over truncation parameter $\delta \leq \Theta\left(\frac{1}{\Delta k}\right)^{k-1}$
- carefully working with hypergraph structure and constructing polymers which are *minimal connected subgraphs* (MCSes) in the incidence graph of H

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Hypergraph matching polytope

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- crucially uses a combination of our technique with that of [Bencs-Regts24]
- Kotecký-Preiss criterion analysis requires a hypergraph generalization of broken circuit theory for graphs [Whitney32] [Tutte54]

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- We obtain an FPTAS for polytope volume using the cluster expansion technique
- We develop an FPTAS for the truncation of the hypergraph matching polytope, analyzed using a novel generalization of broken circuit theory for hypergraphs

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  $f(\Delta)n^c$ running time instead of current $n^{f(\Delta)}$?

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- Relaxing constant-degree assumption?

  perhaps on special classes of graphs like expander, random, bipartite graphs?

Thank you!